$$\text{①}\begin{cases}\Delta u = \cos\varphi & r>3,\ \varphi\in[0,2\pi) \\ u(3,\varphi) = 3\sin 2\varphi & \varphi\in[0,2\pi)\end{cases}$$

~~u(r,φ)~~ $\Rightarrow$

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} = \cos\varphi$$

~~Вз~~ Попробуем искать частное решение в виде

$A r^2\cos\varphi$: $\quad (2r^2)' = 4r$

$$\frac{1}{r}A\cancel{\cos\varphi}(r\cdot 2r)' + \frac{1}{\cancel{r^2}}A\cancel{r^2}(-\cancel{\cos\varphi}) = \cos\varphi$$

$$4A \cancel{+} - A = 1 \Rightarrow A = \frac{1}{3} \Rightarrow$$

$$u_{част} = \frac{r^2}{3}\cos\varphi$$

Пусть $v = u - u_{част}$

$$\begin{cases}\Delta v = 0 & r>3,\ \varphi\in[0,2\pi) \\ v(3,\varphi) = 3\sin 2\varphi - \frac{3^2}{3}\cos\varphi & \varphi\in[0,2\pi)\end{cases}$$

Вне круга $\Rightarrow$

$$v(r,\varphi) = A + \sum_{n=1}^{\infty} r^{-n}(a_n\cos n\varphi + b_n\sin n\varphi) =$$

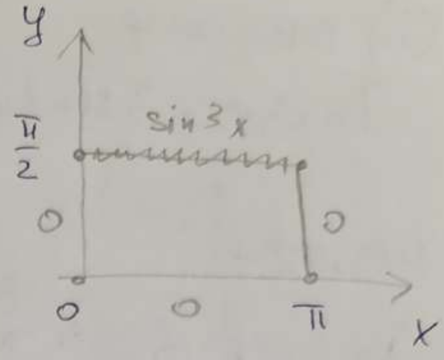
$$= 3\sin 2\varphi - 3\cos\varphi$$

В силу л.н. $1, \cos n\varphi, \sin n\varphi$ можем приравнять почленно.

$$v(r,\varphi) = \frac{27\sin 2\varphi}{r^2} - \frac{9\cos\varphi}{r}$$

$$\begin{cases}-3 = \frac{1}{3}\cdot a_n \Rightarrow \\ 3 = \left(\frac{1}{3}\right)^2\cdot b_n\end{cases} \Rightarrow \boxed{u(r,\varphi) = \frac{27\sin 2\varphi}{r^2} - \frac{9\cos\varphi}{r} + \frac{r^2\cos\varphi}{3}}$$

② $$\begin{cases} \Delta u = 0, & 0 < x < \pi, \quad 0 < y < \frac{\pi}{2} \\ u|_{x=0} = 0 & u|_{x=\pi} = 0 \\ u|_{y=0} = 0 & u|_{y=\frac{\pi}{2}} = \sin 3x \end{cases}$$



$\exists\ u(x,y) = X(x)\,Y(y)$

$\Delta u = X''(x)\,Y(y) + X(x)\,Y''(y) = 0 \quad \Big|:XY$

$X \equiv 0$ или $Y \equiv 0$ не явл-ся решениями в силу $u|_{y=\frac{\pi}{2}} = \sin 3x$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{cases} \Rightarrow \quad \lambda_n = n^2, \quad X_n = \sin nx$$

$$u(x,y) = \frac{sh(3y)}{sh(\frac{3\pi}{2})} \sin 3x.$$

$$u = \sum_{n=1}^{\infty} X_n Y_n. \quad u|_{y=0} = \sum_{n=1}^{\infty} X_n(x)\,Y_n(0) = 0$$

$$u|_{y=\frac{\pi}{2}} = \sum_{n=1}^{\infty} X_n(x)\,Y_n\left(\frac{\pi}{2}\right) = \sin 3x$$

$$\Rightarrow \begin{cases} Y_3'' - \lambda_3 Y_3 = 0 \\ Y_3(0) = 0 \\ Y_3\left(\frac{\pi}{2}\right) = 1 \end{cases} \quad \text{и} \quad \begin{cases} Y_n'' - \lambda_n Y_n = 0 \\ Y_n(0) = 0 \\ Y_n\left(\frac{\pi}{2}\right) = 0 \end{cases} \quad \forall n \neq 3$$

$\Rightarrow Y_n(y) \equiv 0 \quad n \neq 3$

$\exists\ Y_3(y) = e^{ky}.\quad \lambda_3 = 3^2 = 9$

$k^2 - 9 = 0, \quad k = \pm 3$

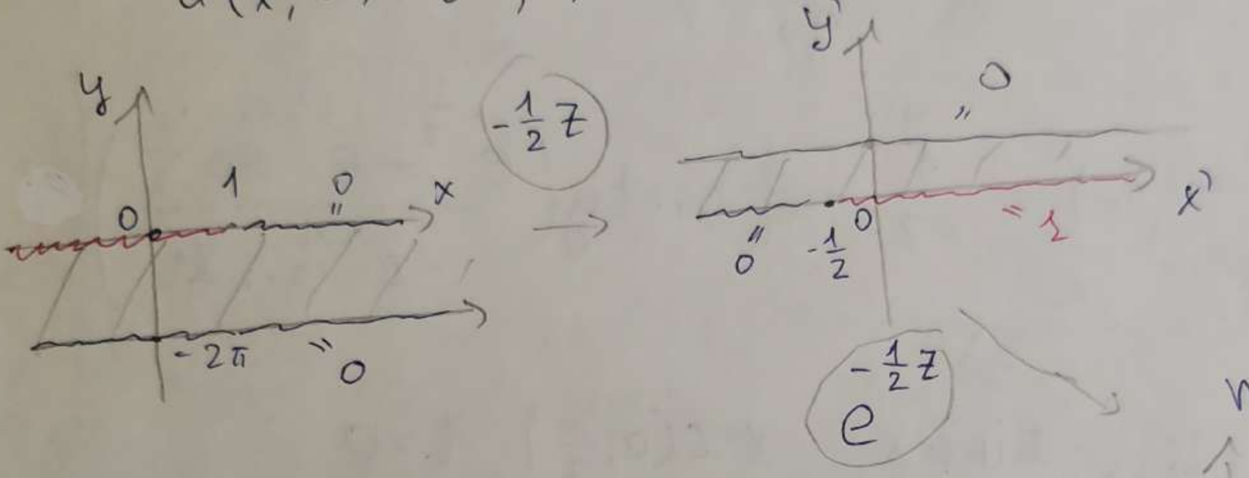
$Y_{3\,общ}(y) = C_1 e^{3y} + C_2 e^{-3y}$

$Y_3(0) = C_1 + C_2 = 0 \Rightarrow C_2 = -C_1$;

$Y_3\left(\frac{\pi}{2}\right) = C_1\left(e^{\frac{3\pi}{2}} - e^{-\frac{3\pi}{2}}\right) = 1 = \tilde{C}_1\, sh\left(\frac{3\pi}{2}\right) \Rightarrow$

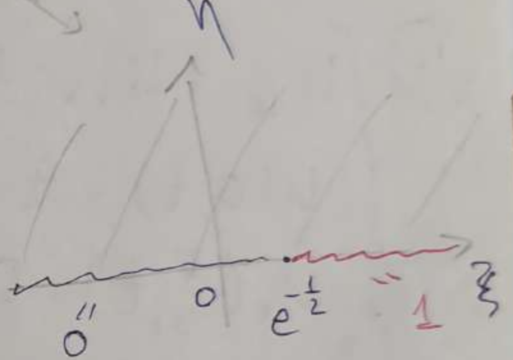
$Y_3(y) = \dfrac{sh(3y)}{sh(\frac{3\pi}{2})}$

(3) $\begin{cases} \Delta u = 0 & -\infty < x < +\infty, \ -2\pi < y < 0 \\ u(x, -2\pi) = 0 \\ u(x, 0) = 1, & x < 1 \\ u(x, 0) = 0, & x \geq 1 \end{cases}$



$$w = e^{-\frac{1}{2}z}$$

$$z = x + iy \Rightarrow w = e^{-\frac{1}{2}(x+iy)} =$$



$$e^{-\frac{1}{2}x} \cdot e^{i(-\frac{1}{2}y)} = e^{-\frac{1}{2}x}\left(\cos\frac{1}{2}y - \sin\frac{1}{2}y\right)$$

$$\xi_0 = e^{-\frac{1}{2}x_0} \cos\frac{y_0}{2}$$

$$\eta_0 = e^{-\frac{1}{2}x_0}\left(-\sin\frac{y_0}{2}\right)$$

$\exists\ u(z) = v(w)$

$$\begin{cases} \Delta v = 0 \\ v|_{\eta=0} = \begin{cases} 1, & \xi > e^{-\frac{1}{2}} \\ 0, & \xi \leq e^{-\frac{1}{2}} \end{cases} = F(\xi) \end{cases}$$

Решение на полуплоскости:

$$v(w_0) = \frac{\eta_0}{\pi}\int_{-\infty}^{\infty} \frac{F(\xi)\,d\xi}{(\xi-\xi_0)^2+\eta_0^2} = \frac{\eta_0}{\pi}\int_{e^{-\frac{1}{2}}}^{\infty} \frac{d\xi}{(\xi-\xi_0)^2+\eta_0^2} =$$

$$= \frac{\eta_0}{\pi}\int_{e^{-\frac{1}{2}}}^{\infty} \frac{1}{\eta_0^2}\frac{d(\xi-\xi_0)}{1+\left(\frac{\xi-\xi_0}{\eta_0}\right)^2} = \frac{1}{\pi}\int_{e^{-\frac{1}{2}}}^{\infty} \frac{d\left(\frac{\xi-\xi_0}{\eta_0}\right)}{1+\left(\frac{\xi-\xi_0}{\eta_0}\right)^2} \ominus$$

$$\circledequal \ \frac{1}{\pi} \operatorname{arctg} \frac{\xi - \xi_0}{\eta_0} \Bigg|_{\xi = e^{-\frac{1}{2}}}^{\xi = +\infty} =$$

$$\frac{1}{2} - \frac{1}{\pi} \operatorname{arctg}\left(\frac{e^{-\frac{1}{2}} - \xi_0}{\eta_0}\right) = v(w_0)$$

Тогда $\boxed{u(z_0) = \frac{1}{2} - \frac{1}{\pi} \operatorname{arctg}\left(\frac{e^{-\frac{1}{2}} - e^{-\frac{1}{2}x_0} \cos \frac{y_0}{2}}{-e^{-\frac{1}{2}x_0} \sin \frac{y_0}{2}}\right)}$

④ $\begin{cases} u_{tt} = u_{xx} - \sin 3x, \quad x \in (0, \frac{\pi}{2}), \ t > 0 \\ u(0,t) = t \\ u_x(\frac{\pi}{2}, t) = 0 \\ u(x,0) = 2\sin 3x - \sin x \\ u_t(x,0) = \sin x + 1 \end{cases}$

Обнуляем гр. условие: $u(x,t) = v(x,t) + t$

$\begin{cases} v_{tt} = v_{xx} - \sin 3x = f(x) \\ v(0,t) = 0 \\ v_x(\frac{\pi}{2}, t) = 0 \\ v(x,0) = 2\sin 3x - \sin x = \varphi(x) \\ v_t(x,0) = \sin x = \psi(x) \end{cases}$ Гр. усл. I – II $\Rightarrow$

$\lambda_n = (2n-1)^2, \ n \in \mathbb{N}$

$X_n = \sin(2n-1)x$

$\Rightarrow$

$f(x), \varphi(x), \psi(x)$ уже разложены по с.ф.

можно искать ответ в виде

$$w(x,t) = A(t) \sin x + B(t) \sin 3x$$

Тогда, подставив в ур-е ~~...~~ колебаний, получаем

$$\begin{cases} A'' \sin x + B'' \sin 3x = -A\sin x - 9B\sin 3x + \sin 3x \\ A(0)\sin x + B(0)\sin 3x = -\sin x + 2\sin 3x \\ A'(0)\sin x + B'(0)\sin 3x = \sin x \end{cases}$$

В силу л.н. $\sin(2n-1)x$ можем приравнять почленно:

$$\begin{cases} A'' = -A \\ A(0) = -1 \\ A'(0) = 1 \end{cases} \Rightarrow \quad A_{общ}(t) = C_1 \cos t + C_2 \sin t$$

$$A(0) = C_1 = -1$$

$$A'(0) = C_2 = 1$$

$$\Rightarrow A(t) = \sin t - \cos t$$

$$\begin{cases} B'' = -9B + 1 \\ B(0) = 2 \\ B'(0) = 0 \end{cases} \Rightarrow \quad B_{одн}(t) = C_1 \cos 3t + C_2 \sin 3t$$

$$B_{част}(t) = a:$$

$$0 = -9a + 1 \Rightarrow a = \frac{1}{9}$$

$$B(0) = C_1 + \frac{1}{9} = 2 \Rightarrow C_1 = \frac{17}{9}$$

$$B'(0) = 3C_2 = 0 \Rightarrow C_2 = 0$$

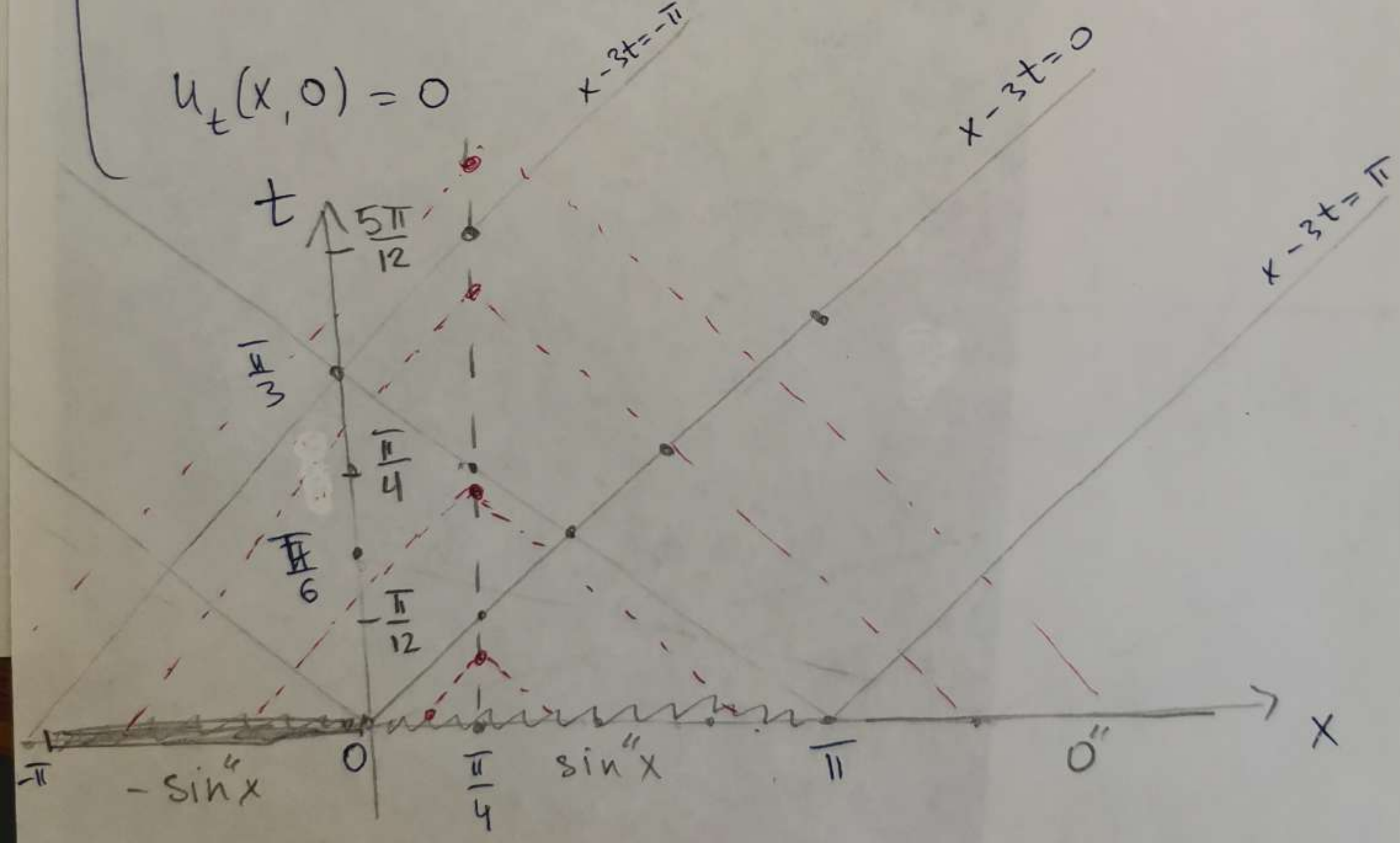
$$v(x,t) = (\sin t - \cos t)\sin x + \left(\frac{17}{9}\cos 3t + \frac{1}{9}\right)\sin 3x$$

$$\boxed{u(x,t) = t + (\sin t - \cos t)\sin x + \left(\frac{17}{9}\cos 3t + \frac{1}{9}\right)\sin 3x}$$

5.
$$\begin{cases} u_{tt} = 9u_{xx} & x>0,\ t>0 \\ u_x(0,t) = 0 \\ u(x,0) = \begin{cases} \sin x, & x \in [0,\pi) \\ 0, & x \notin [0,\pi) \end{cases} \\ u_t(x,0) = 0 \end{cases}$$

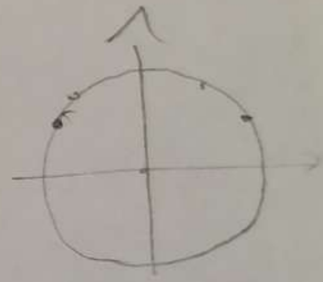
Р-м вспомогательную задачу

$$\begin{cases} u_{tt} = 9u_{xx} \\ u(x,0) = \begin{cases} \sin x, & x \in [0,\pi] \\ -\sin x, & x \in [-\pi,0) \\ 0, & x \notin [-\pi,\pi) \end{cases} = \Phi(x) \\ u_t(x,0) = 0 \end{cases}$$



$$u\left(\frac{\pi}{4},t\right) = \frac{\Phi\left(\frac{\pi}{4}-3t\right)+\Phi\left(\frac{\pi}{4}+3t\right)}{2}$$

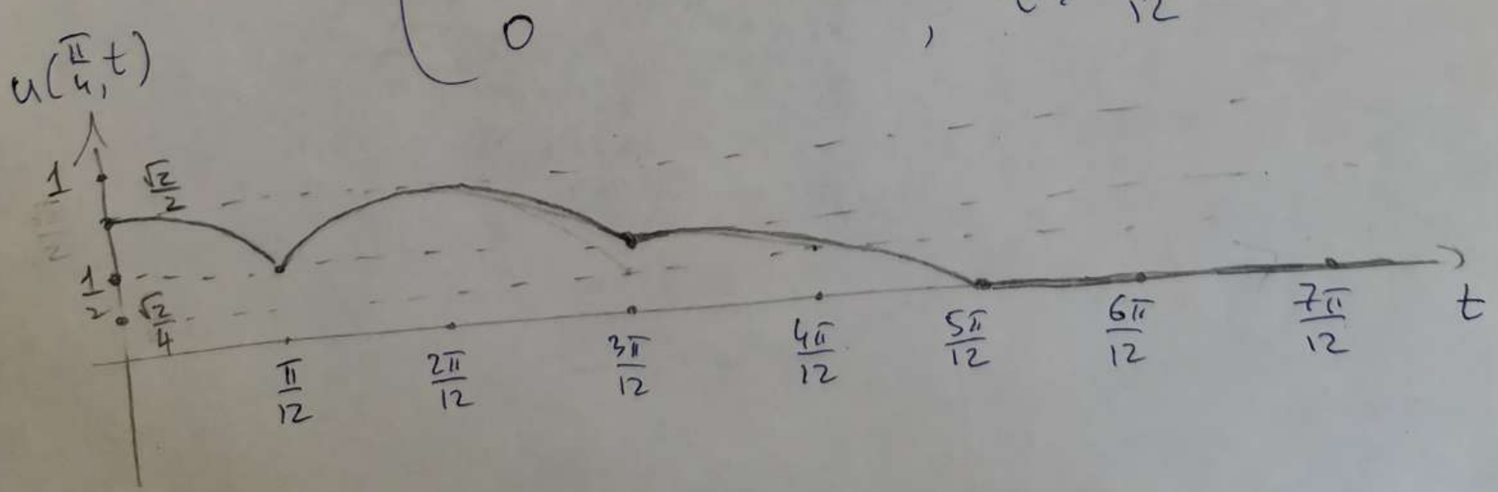
$$u\left(\frac{\pi}{4},t\right)=\begin{cases}\dfrac{\sin\left(\frac{\pi}{4}-3t\right)+\sin\left(\frac{\pi}{4}+3t\right)}{2}, & 0\le t<\frac{\pi}{12}\\[2ex] \dfrac{-\sin\left(\frac{\pi}{4}-3t\right)+\sin\left(\frac{\pi}{4}+3t\right)}{2}, & \frac{\pi}{12}\le t<\frac{3\pi}{12}\\[2ex] \dfrac{-\sin\left(\frac{\pi}{4}-3t\right)}{2}, & \frac{3\pi}{12}\le t<\frac{5\pi}{12}\\[2ex] 0, & t\ge\frac{5\pi}{12}\end{cases}$$



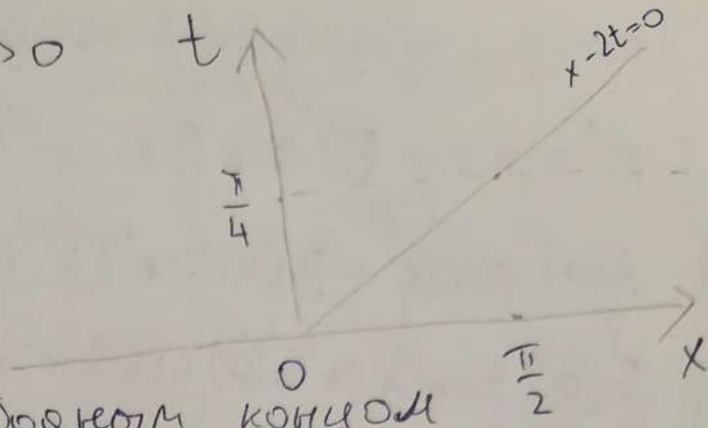
$$\sin\left(\frac{\pi}{4}-3t\right)=\frac{\sqrt{2}}{2}(\cos 3t-\sin 3t)$$

$$\sin\left(\frac{\pi}{4}+3t\right)=\frac{\sqrt{2}}{2}(\cos 3t+\sin 3t)$$

$$u\left(\frac{\pi}{4},t\right)=\begin{cases}\frac{\sqrt{2}}{2}\cos 3t, & 0\le t<\frac{\pi}{12}\\[1ex] \frac{\sqrt{2}}{2}\sin 3t, & \frac{\pi}{12}\le t<\frac{3\pi}{12}\\[1ex] \frac{1}{2}\sin\left(3t-\frac{\pi}{4}\right), & \frac{3\pi}{12}\le t<\frac{5\pi}{12}\\[1ex] 0, & t\ge\frac{5\pi}{12}\end{cases}$$



⑥ $\begin{cases} u_{tt} = 4u_{xx} & x>0,\ t>0 \\ u_x(0,t) = 0 \\ u(x,0) = \sin x \\ u_t(x,0) = 2\cos x \end{cases}$



$u_x(0,t) = 0$ – колебания со свободным концом

Р-м вспомогательную задачу

$$\begin{cases} u_{tt} = 4u_{xx} & x \in \mathbb{R},\ t>0 \\ u(x,0) = \varphi(x) = \begin{cases} \sin x, & x \geq 0 \\ -\sin x, & x < 0 \end{cases} \\ u_t(x,0) = \psi(x) = 2\cos x & x \in \mathbb{R} \end{cases}$$

$$u(x,t) = \frac{\varphi(x-2t) + \varphi(x+2t)}{2} + \frac{1}{2\cdot 2}\int_{x-2t}^{x+2t} \psi(\alpha)\, d\alpha$$

$$u(x,t) = \begin{cases} \dfrac{\sin(x+2t) - \sin(x-2t)}{2} + \dfrac{1}{2}\displaystyle\int_{x-2t}^{x+2t} \cos\alpha\, d\alpha, & x \in [0, 2t) \\ \dfrac{\sin(x+2t) + \sin(x-2t)}{2} + \dfrac{1}{2}\displaystyle\int_{x-2t}^{x+2t} \cos\alpha\, d\alpha, & x \geq 2t \end{cases}$$

$$u\left(x,\frac{\pi}{4}\right) = \begin{cases} \underbrace{\dfrac{\sin\left(x+\frac{\pi}{2}\right) - \sin\left(x-\frac{\pi}{2}\right)}{2}}_{\cos x} + \dfrac{1}{2}\displaystyle\int_{x-\frac{\pi}{2}}^{x+\frac{\pi}{2}} \cos\alpha \, d\alpha \,, & x < \frac{\pi}{2} \\ \underbrace{\dfrac{\sin\left(x+\frac{\pi}{2}\right) + \sin\left(x-\frac{\pi}{2}\right)}{2}}_{0} + \underbrace{\dfrac{1}{2}\displaystyle\int_{x-\frac{\pi}{2}}^{x+\frac{\pi}{2}} \cos\alpha \, d\alpha}_{\frac{1}{2}\sin\alpha\Big|_{x-\frac{\pi}{2}}^{x+\frac{\pi}{2}} = \cos x} \,, & x \geq \frac{\pi}{2} \end{cases}$$

$$u\left(x,\frac{\pi}{4}\right) = \begin{cases} 2\cos x \,, & x < \frac{\pi}{2} \\ \cos x \,, & x \geq \frac{\pi}{2} \end{cases}$$